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Inflation tolerance ranges in the New Keynesian model

Hervé Le Bihan[1], Magali Marx[2] & Julien Matheron[3]

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Abstract

A number of central banks in advanced countries use ranges, or bands, around their inflation target to formulate their monetary policy strategy. The adoption of such ranges has been proposed by some policymakers in the context of the Fed and the ECB reviews of their strategies. Using a standard New Keynesian macroeconomic model, we analyze the consequences of tolerance range policies, characterized by a stronger reaction of the central bank to inflation when inflation lies outside the range than when it is close to the target, ie the central value of the band. We show that (i) a tolerance band should not be a zone of inaction: the lack of reaction within the band endangers macroeconomic stability and leads to the possibility of multiple equilibria; (ii) the trade-off between the reaction needed outside the range versus inside seems unfavorable: a very strong reaction, when inflation is far from the target, is required to compensate a moderately lower reaction within tolerance band; (iii) these results, obtained within the framework of a stylized model, are robust to many alterations, in particular allowing for the zero lower bound.[4]

Keywords: Monetary policy; inflation ranges; inflation bands; ZLB; endogenous regime switching.

JEL classification: E31, E52, E58.

[1] Banque de France, herve.lebihan@banque-france.fr
[2] Banque de France, magali.marx@banque-france.fr
[3] Banque de France, julien.matheron@banque-france.fr
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NON-TECHNICAL SUMMARY

In the context of recent Monetary Policy Strategy Reviews in the US and the euro area, there has been a renewed interest for the notion of an inflation "tolerance band", as a possible element to include in a revamped monetary policy framework.

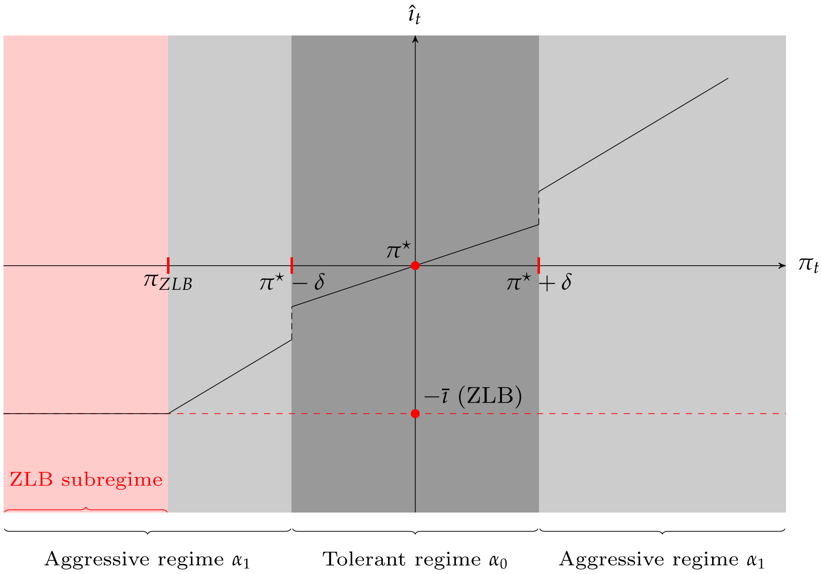
In spite of the recurrence of debates about inflation ranges in discussions of monetary policy frameworks, the analytical literature is surprisingly scarce on the properties of such set-ups. To our knowledge, there has not been a systematic attempt to study such policies in the New-Keynesian model - arguably by now the most standard set-up for monetary policy analysis. The aim of this paper is to contribute to filling this gap.

In this paper, we interpret the notion of "tolerance bands" as the central bank having a more aggressive response to inflation when inflation is outside the "tolerance band" than when it lies within the band. (Other interpretations are briefly discussed in the paper). This notion appears as an extension of the concept of indifference band, in which the degree of indifference to inflation, within the band, is not complete and may take on alternative values.

This approach is particularly relevant in the context of the euro area debate. Indeed, back in the initial stage of the euro, prior to 2003, the framework of the Eurosystem could be interpreted as an "indifference range". More recently, some policy proposals for inflation "tolerance bands" were calling for a higher degree of flexibility in monetary policy. The idea was that, whenever inflation is reasonnably close to the target - and reflecting secondary objectives for monetary policy, such as financial stability concerns-, a given deviation of inflation from target may call for a smaller reaction than otherwise warranted by a strict inflation targeting.

We conduct a theoretical and quantitative evaluation of inflation "tolerance bands" in a standard New Keynesian model. We reach the following main conclusions. First, the state-dependent policy rule we consider captures a notion of policy patience. Indeed, after an inflationnary shock, whenever inflation lies within the bands (i.e, is close to the target), the tolerance range setup will allow for a slower convergence back to the inflation target. Second, to achieve macroeconomic stability, an active monetary policy rule is needed even when inflation lies within the tolerance range. This provides a formal basis for claims by policymakers (e.g. in Coeuré, 2019) that a tolerance range should not be interpreted as an inaction range. Third, there is a quantitative trade-off between the degree of activism within the inflation range vs. that outside the range. This trade-off proves to be quantitatively unfavorable. In effect, for the central bank to stabilize inflation over the cycle, lowering the systematic reaction of inflation within the tolerance range requires a large increase in the systematic reaction outside the tolerance range. Moreover, along this "iso-variance" curve, the volatility of the nominal interest rate increases with the difference between the degrees of reaction to inflation inside and outside the tolerance range. Thus, it is possible to maintain a constant level of inflation volatility by adopting a slightly more lenient policy within the tolerance range and a strongly more aggressive policy outside the range, but this comes at the cost of a substantial increase in the volatility of the nominal interest rate. Fourth, when the risk of interest rates hitting the Zero Lower Bound (or an Effective Lower Bound) is taken into account, while the overall stabilization performances are worsened, the trade-off involved by tolerance ranges is broadly unchanged.

The interest rate rule with inflation tolerance ranges



Bandes de tolérance à l'inflation dans un modèle néo- keynésien

Résumé

De nombreuses banques centrales dans les pays avancés ont recours au concept d'intervalle ou de bande autour de leur cible d'inflation pour formuler leur stratégie de politique monétaire. L'adoption d'un tel concept d'intervalle a été proposée par certains banquiers centraux dans le cadre des revues stratégiques de la Fed et de la BCE. Dans le cadre d'un modèle macroéconomique néo-keynésien standard, nous analysons les conséquences de politiques fondées sur un tel intervalle de tolérance, caractérisées par une réaction de la banque centrale à l'inflation plus forte quand l'inflation est en dehors de l'intervalle que lorsqu'elle est proche de la cible (le centre de l'intervalle). Nous montrons que (i) il n'est pas souhaitable qu'une bande de tolérance soit une zone d'inaction : l'absence de réaction dans l'intervalle menace la stabilité macroéconomique et conduit à la possibilité d'équilibres multiples ; (ii) l'arbitrage entre la réaction requise en dehors de l'intervalle et celle à l'intérieur semble défavorable : il faut une réaction très forte loin de la cible pour compenser une réaction légèrement plus faible à l'intérieur de l'intervalle de tolérance ; (iii) ces résultats, obtenus dans le cadre d'un modèle stylisé, sont robustes à de nombreuses variantes, notamment la prise en compte de la borne inférieure pour le taux d'intérêt.

Mots-clés : bande d'inflation, politique monétaire ; borne inférieure du taux d'intérêt ; changement de régime endogène.

1. Introduction

Against the backdrop of the Monetary Policy Strategy Reviews in the US and the euro area, there has been a renewed interest for the notion of inflation range as an element of the monetary policy framework.[1] Brainard (2020), Coeuré (2019), Knot (2019), Rosengren (2018) are relevant mentions by monetary policymakers. In spite of the recurrence of debates about inflation ranges in discussions of monetary policy frameworks, the analytical literature is surprisingly scarce on the properties of such set-ups. To our knowledge, there has not been a systematic attempt to study such policies in the New-Keynesian model – arguably by now the most standard set-up for monetary policy analysis. The aim of this paper is to contribute to filling this gap.

One challenge in this endeavour is that the notion of inflation ranges has taken on different guises, as evidenced by the different wording and communication details regarding inflation ranges across central banks and by the variety in the terminology related to inflation ranges.[2] Chung, Doyle, Hebden, and Siemer (2020) have recently put forward a taxonomy, distinguishing between three broad concepts: "uncertainty ranges", "operational ranges", and "indifference ranges".

In this paper, we interpret the notion of "inflation tolerance range" as the central bank having a more aggressive response to inflation when inflation is outside the "tolerance range" than when it lies within the range. In terms of the taxonomy by Chung et al. (2020), the notion of tolerance ranges we consider is thus an extension of the indifference ranges, in which the degree of indifference to inflation, within the range, is not complete and may take on alternative values. We embed this state-dependent formulation of monetary policy in an otherwise standard New Keynesian model to conduct a theoretical and quantitative evaluation of inflation "tolerance ranges".

This approach is particularly relevant in the context of the euro area debate. Indeed, back in the initial stage of the euro, prior to 2003, the framework of the Eurosystem could be interpreted as an "indifference range".[3] More recently, proposals for inflation "tolerance ranges", discussed e.g. by Coeuré (2019) and Knot (2019), have strived for a higher degree of flexibility in monetary policy. They suggest that whenever inflation is reasonnably close to the target, and reflecting secondary objectives for monetary policy, such as financial stability concerns, the deviation of inflation from target may call for a smaller reaction than

[1]Prominent examples of central bank using such ranges in their monetary policy frameworks are the Bank of Canada and the Bank of Sweden. The Bank of Canada "aims to keep total CPI inflation at the 2 per cent midpoint of a target range of 1 to 3 per cent over the medium term" and the Bank of Sweden uses a "variation band of 1-3 percent" around the 2 percent inflation target.

[2]Examples of wording are "control range", "variation bands", "inflation ranges". In this paper, we will use indifferently the terms of "range" and "band".

[3]See Rostagno, Altavilla, Carboni, Lemke, Motto, Saint Guilhem, and Yiangou (2019) for a discussion.

otherwise warranted by a strict inflation targeting.[4] Under this set-up, we reach the following main conclusions.

First, the state-dependent policy rule we consider captures a notion of policy patience. Indeed, after an inflationary shock, whenever inflation lies within the bands (i.e, is close to the target), the tolerance range setup will allow for a slower convergence back to the inflation target.

Second, to achieve macroeconomic stability, an active monetary policy rule is needed even when inflation lies within the tolerance range. This provides a formal basis for claims by policymakers that a tolerance range should not be interpreted as an inaction range.

Third, there is a quantitative trade-off between the degree of activism within the inflation range vs. that outside the range. This trade-off proves to be quantitatively unfavorable. In effect, for the central bank to keep the variance of inflation at a given level, lowering the systematic reaction of inflation within the tolerance range requires a large increase in the systematic reaction outside the tolerance range. Moreover, along this iso-variance curve, the volatility of the nominal interest rate increases with the difference between the degrees of reaction to inflation inside and outside the tolerance range. Thus, it is possible to maintain a constant level of inflation volatility by adopting a slightly more lenient policy within the tolerance range and strongly more aggressive policy outside the range, but this comes at the cost of a substantial increase in the volatility of the nominal interest rate.

Fourth, when the risk of interest rates hitting the Zero Lower Bound (or an Effective Lower Bound) is taken into account, while the overall stabilization performances are worsened, the trade-off involved by ranges is broadly unchanged. Besides the results on the substance, one contribution of ours is to illustrate how resorting to recent techniques for solving endogenous switching models helps alleviate the technical difficulty raised by inflation bands, which introduce non-linearities and non-differentiabilities in the policy rule. Such technical difficulties may to some extent explain the lack of previous analyses on such policies.

Our paper is related to a scarce academic literature on inflation ranges. Castelnuovo, Rodriguez-Palenzuela, and Nicoletti Altimari (2003), Ehrmann (2021), Grosse-Steffen (2021) carry out empirical comparisons of the performance of countries with and without an inflation band. Orphanides and Wieland (2000) have provided possible motivations, in a model with backward-looking IS and Phillips curves, based on: (i) non quadratic zone loss function, or (ii) a non-linear Phillips curve. By contrast, our analysis is only positive, as we do not derive inflation tolerance ranges as an optimal policy.[5] On the other hand, our set-up, incorporating explicit agents expectations, allows us inter alia to rely on the current standard model for monetary policy analysis, and to discuss determinacy issues arising with tolerance

[4]Acknowledgedly, our approach leaves aside the use of inflation ranges as a mere communication device by central banks in the form of "uncertainty ranges", as well as the use of bands to engineer temporary inflation over-shooting as advocated by some policy makers in the recent US policy debate (Brainard, 2020).

[5]Note the Taylor rule assumed in our model has the same piecewise form as derived as an optimal policy in that paper

ranges, that are absent in backward looking models. Mishkin and Westelius (2008) provide an alternative rationalization for inflation ranges based on inflation contracts in a Barro-Gordon (1983) set-up with a “new classical” Phillips curve, in which the preferences of the central bank differ from what is socially optimal. Svensson (1997) discusses the use of bands for communication and accountability purposes. Chung et al. (2020), using simulations of the FRB/US model, show that the ability to stabilize the economy, facing various shocks, is worsened when the central bank adopts an “indifference range” policy. These results are consistent with our findings, obtained using a smaller model. The set-up we use allows us to provide some analytical results and undertake a systematic analysis of inflation range policies. In particular we shed light on which parameters affect the relative performance of inflation ranges and focal points objectives. Using, as we do, a model close to the baseline New Keynesian model, Bianchi, Melosi, and Rottner (2019) focus on asymmetric inflation band as a strategy to handle the deflation bias related to the zero lower bound. We mainly focus on symmetric inflation ranges, but take on board the consequences of the ZLB for the trade-off we study.

The paper is structured as follows. Section 2 presents our baseline model, and the formalisation of inflation tolerance ranges. Section 3 derives some general results regarding inflation stabilization properties under inflation range policies Section 4 investigates the quantitative trade-off between within the range and outside the range activism, using an extended model featuring additional dynamics and closer to models used in actual policy analysis. Section 5 studies the implication of the incidence of the Zero (or Effective) Lower Bound for inflation range policies.

2. The simple New Keynesian model with inflation tolerance range

We consider the following model, a version of the standard three equation New Keynesian (NK) model (see e.g. Galí 2015). The model consists of the following three equations:

$$x_t = \mathbb{E}_t\{x_{t+1}\} - \sigma(\hat{\imath}_t - \mathbb{E}_t\{\hat{\pi}_{t+1}\}) + \epsilon_t^x \tag{1}$$

$$\hat{\pi}_t = \beta\mathbb{E}_t\{\hat{\pi}_{t+1}\} + \kappa x_t + \epsilon_t^\pi \tag{2}$$

$$\hat{\imath}_t = \phi_r\hat{\imath}_{t-1} + (1-\phi_r)(\alpha_{s_t}\hat{\pi}_t + \phi_x x_t) \tag{3}$$

$$\alpha_{s_t} = \alpha_0 \quad \text{if} \quad |\hat{\pi}_{t-1}| < \delta, \quad \alpha_1 \quad \text{if} \quad |\hat{\pi}_{t-1}| > \delta$$

where $\mathbb{E}_t\{\cdot\}$ denotes expectations conditioned on the information set available as of time t, x_t is the output gap, π_t is the quarterly inflation rate, and $\hat{\pi}_t \equiv \pi_t - \pi^\star$ is the deviation of inflation from the central bank target $\pi^\star$. The gap between the nominal interest rate i_t and its steady-state value $\bar{\imath}$ is denoted by $\hat{\imath}_t \equiv \imath_t - \bar{\imath}$. The term ϵ_t^x corresponds to a demand shock and ϵ_t^π corresponds to a cost-push shock.

Moreover, we assume that ϵ_t^x and ϵ_t^π satisfy

$$\epsilon_t^x = \rho_x \epsilon_{t-1}^x + \nu_t^x, \qquad \epsilon_t^\pi = \rho_\pi \epsilon_{t-1}^\pi + \nu_t^\pi$$

where ν_t^π and ν_t^x are assumed i.i.d with variances $\sigma_x^2 = \mathbb{E}\{(\nu_t^x)^2\}$ and $\sigma_\pi^2 = \mathbb{E}\{(\nu_t^\pi)^2\}$.

The first two equations are the familiar IS and Phillips curves, while the third equation is the interest rate rule followed by the central bank. The model explicitly features a (strictly positive) inflation target: while its level will not be relevant in our baseline analysis, it is useful to introduce it explicitly in connection with actual policy discussions, and in anticipation of the introduction of the zero lower bound. The Phillips curve, the middle equation in (2), assumes firms not re-optimizing their price are able to index them to steady-state inflation. The only difference from the standard NK framework is that the policy rule parameter α_{s_t} is a function of the regime realized at t, s_t . We will hereafter refer to the central bank interest-rate rule as the Markov-Switching Taylor rule.

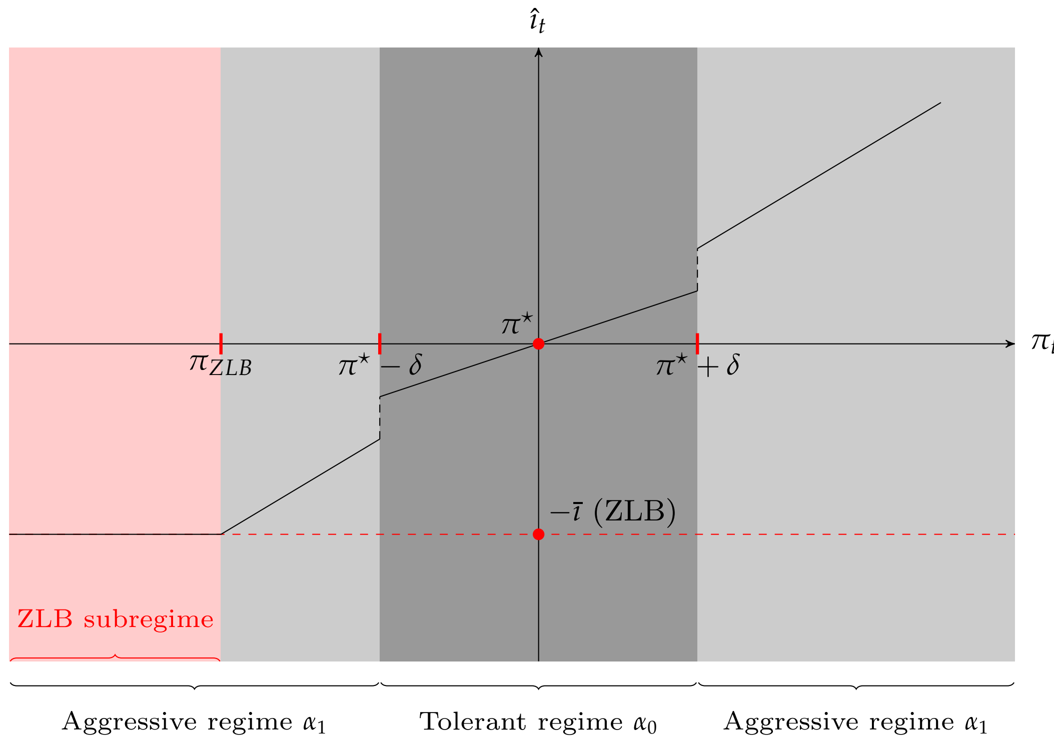
The policy regime s_t, in turn, is endogenously determined according to whether inflation lies within a tolerance range. This tolerance range is assumed to be a symmetric region $\pm\delta$ around the inflation target ($\pi^\star$). Thus the bandwidth of the tolerance range is 2δ.[6] Whenever last period's inflation falls outside the tolerance range ($\pi_{t-1} \notin [\pi^\star - \delta; \pi^\star + \delta]$), the policy rule will switch to the more aggressive regime $s_t = 1$, resulting in the inflation parameter $\alpha_1 > \alpha_0$ being used in the rule. By contrast, whenever $\pi_{t-1} \in [\pi^\star - \delta; \pi^\star + \delta]$, that is when last period's inflation falls within the tolerance range, the policy rule will revert to a less aggressive rule characterized by parameter α_0. Note that our set-up obviously encompasses the standard NK set-up, which obtains when $\delta = 0$.

The regime-switching policy rule is illustrated in Figure 1, in the particular case of no reaction to output, and no persistence, i.e. $\hat{\imath}_t = \alpha_{s_t}\hat{\pi}_t$. In the dark gray area, corresponding to the tolerance range $[\pi^\star - \delta, \pi^\star + \delta]$, the policy rule is governed by the parameter α_0. The light gray areas correspond to the more aggressive regime $\alpha_1 > \alpha_0$. Note that on the left part of the figure, the pink area delineates the Zero Lower Bound regime. In this section, we abstract from this extra regime and defer a discussion of the ZLB until section 5. For later reference, note however that the higher δ, and the higher α_1, the thinner the left mild gray area.

It is worth underlying that the policy rule considered here defines a tolerance range around a focal point which is the inflation target $\pi^\star$. In effect, having both a range and a central inflation target is the most common practice among central banks that use an inflation range, though in principle and in some actual cases, the central bank may provide only a range without emphasizing a central value (see Grosse-Steffen 2021 for a review of the international evidence) .

[6]While not the focus of our paper, our framework can allow for an asymmetric range, as in Bianchi et al. (2019). As tolerance ranges currently employed by central banks are generally symmetric, we do not develop this case here.

Figure 1. The interest rate rule with inflation tolerance ranges



A feature of our policy rule worth emphasizing is that it entails a discontinuity at the edges of the tolerance band, as Figure 1 makes clear. One could object that an alternative approach to modeling tolerance ranges would have entailed a continuous representation, albeit with possible kinks. Such an alternative specification could be

$$\hat{\imath}_t = \alpha_0 \hat{\pi}_t \times \mathbb{1}_{\{\pi_{t-1} \in [\pi^\star - \delta; \pi^\star + \delta]\}}$$
$$+ [\alpha_0 \delta + \alpha_1(\hat{\pi}_t - \delta)] \times \mathbb{1}_{\{\pi_{t-1} > \pi^\star + \delta\}} + [-\alpha_0 \delta + \alpha_1(\hat{\pi}_t + \delta)] \times \mathbb{1}_{\{\pi_{t-1} < \pi^\star - \delta\}}$$

where, $\mathbb{1}_{\{.\}}$ is an indicator function and, again, inertia and the output gap are disregarded for illustrative purpose.

This specification would remove the discontinuity at the edges of the band, as illustrated in Figure 12 in the Appendix. We do not pursue this approach here firstly since, in our view, emphasis on a range is best captured by a discontinuous reaction function. In addition, under the continuous rule above, multiplicity of equilibria is a stronger concern, making it less appealing (see Appendix D for further details).

The tolerance range induces a non-linearity in the model, which raises a serious computational challenge. Given the type of exercises and the extended model that we consider in the next section, a global non-linear solution is not feasible. To mitigate this issue, we consider a smooth transition probability between the two regimes and resort to the endogenous regime-switching approach developed by Barthélemy and

Marx (2017). This allows us to get explicit determinacy conditions and to solve and simulate the model quickly.

Formally, we assume that the central bank switches from regime s to regime 0, associated with a low reaction to inflation (i.e. $s_t = 0$), for all $s \in \{0, 1\}$, with probability $p_{s,0}(\hat{\pi}_{t-1})$.[7] The latter is explicitly a function of last period's inflation.[8] The perturbation approach requires that the transition probability be sufficiently smooth in $\hat{\pi}$ (formally it is assumed to be twice continuously differentiable). Finally, the probability to stay inside the band is equal to 1 at the steady state, and the first-derivative of this probability is zero at the steady state. This means that the probability is locally quadratic at the steady state. All the assumptions are summarized in Appendix A, and the specification we retain for $p_{s,0}$ will be described hereafter (in (7)).

3. Analytical results: sufficient and necessary conditions for determinacy

In this section we derive stability conditions for the model, in the spirit of the Taylor principle.[9] A maintained assumption is that outside the tolerance range the Taylor principle is fulfilled, that is

$$\alpha_1 + \frac{(1-\beta)\phi_x}{\kappa} > 1.$$

This is the same condition as in the basic New Keynesian model, see Woodford (2011). Note that in the case $\delta \to 0$, i.e. when the inflation tolerance range arbitrarily narrows, the model collapses to the standard New Keynesian model. In this case, the Taylor principle is a condition for stability (see Woodford 2011, chapter 4).

3.1. A necessary condition for determinacy: Activism within the tolerance band.

Our first result establishes that the policy rule followed when inflation lies within the tolerance band is crucial to rule out the existence of sunspot equilibria. We assume that in a neighborhood of the steady state inside the tolerance band, the probability to stay in $s_t = 0$ is 1.

Proposition 1. *If the Taylor principle is not satisfied inside the band, i.e.*

$$\alpha_0 + \frac{(1-\beta)\phi_x}{\kappa} < 1$$

there exist stationary sunspot equilibria arbitrarily close to the steady state.

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[7] Note that, given state variables, the probability of staying in (or moving to) regime 0 does not depend on the initial regime.

[8] Under rational expectations as in our model, probabilities depending on contemporaneous state variables to inconsistency problems. For more details, see Appendix A.2 in Barthélemy and Marx (2017)

[9] We focus on local stability around the equilibrium. Under the Zero Lower Bound, multiplicity of equilibria and concerns about global determinacy in the vein of Benhabib, Schmitt-Grohe, and Uribe (2001) may arise. These are discussed in our context in section 5 below.

Proposition 1 is a direct application of the more general Proposition 4, presented in Appendix A along with its proof.

Proposition 1 leads to two remarks. First, it shows that determinacy relies on the degree of reaction within the tolerance band. A normative consequence is that tolerance should not be interpreted as synonymous with inaction. To avoid sunspot equilibria, the Taylor principle should apply within the tolerance range, irrespective of the policy followed when inflation is outside the band. Thus, the central bank cannot just rely on the "threat" of a very strong activism outside the band to tame unwarranted equilibria.

Second, the result is quite general : it only depends on the fact that steady-state inflation does not depend on regimes and lies within the tolerance range. In particular, it applies irrespective of whether the transition between the two policy regimes is modeled as discrete or smooth.

3.2. **Determinacy.** We now give conditions on the central bank's behavior when inflation is inside the band, ensuring equilibrium determinacy.

To establish this, we assume that the transition probability between the two regimes is twice continuously differentiable and that the steady-state probability of transiting from regime 0 to regime 1 is 0. Then:

Proposition 2. *If the Taylor principle is satisfied,*

$$\alpha_0 + \frac{(1-\beta)\phi_x}{\kappa} > 1,$$

then, when shocks are small enough, there exists a unique solution.

As above, Proposition 2 is an application of the more general Proposition 4, which is presented in Appendix.

This proposition establishes that if the Taylor principle applies within the tolerance range, and if the shocks are small enough, then there exists a unique solution for the whole model.

The above two results call for several further comments. First, these two results are in some sense relatively intuitive: they show that the determinacy of the model mainly relies on the local behavior around the steady state, which is driven by what happens when inflation is within in the band. A concrete implication of these results is that an inaction band leads to the existence of multiple equilibria, and that while solving the model (even with a computational approach) may yield a solution, there is no obvious argument to rule out other solutions.

Second, they may appear to contradict the literature on "the long-run Taylor principle" in models with regime-switching monetary policy rules, as exposed for instance in Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009), Barthélemy and Marx (2019). This literature shows that it is neither necessary nor sufficient that the Taylor rule holds in both regimes to ensure determinacy: a sufficiently strong deviation in the "hawkish" regime can compensate for a small deviation from the Taylor principle in the "dovish" regime. However, an important difference is that in the set-up used in these papers, the transition between regimes is exogenous. By contrast, in our model of tolerance ranges, the steady state belongs to one of the regimes (the tolerance range i.e. the "dovish" regime). In the neighborhood of the steady state, the probability to stay in this regime ($s_t = 0$) is 1, and this implies that a reaction $\alpha_0 > 1$ in this regime is needed to ensure determinacy.

Lastly, note that while inaction, or excessive inflation tolerance within the range, leads to sunspots the maximal *size* of those sunspots is all the smaller as the band's length is narrow. While we are not able to provide quantitative bounds on the variance resulting from the sunspots, it is clear that with arbitrarily narrow ranges, the size of sunspots will be arbitrarily small, and the concern about indeterminacy will become near-irrelevant. It is also conceivable from this perspective that some degree of indeterminacy could be tolerable for the central bank.

4. Quantitative properties in an augmented NK model

We now study quantitatively the properties of the NK model with an inflation tolerance range. To perform this analysis, we consider an augmented model with additional dynamics. This model encompasses the simple New Keynesian model studied in Section 2, and comes closer to the empirical Dynamic Stochastic General Equilibrium (DSGE) models used in policy institutions.[10]

4.1. **An augmented model.** The augmented model is as follows:

$$x_t = \frac{1}{1+h}\mathbb{E}_t\{x_{t+1}\} + \frac{h}{1+h}x_{t-1} - \frac{(1-h)\sigma}{1+h}(\hat{\imath}_t - \mathbb{E}_t\{\hat{\pi}_{t+1}\}) + \epsilon_t^x \tag{4}$$

$$\hat{\pi}_t = \frac{\beta}{1+\beta\gamma}\mathbb{E}_t\{\hat{\pi}_{t+1}\} + \frac{\gamma}{1+\beta\gamma}\hat{\pi}_{t-1} + \frac{\kappa}{1+\beta\gamma}[(\omega+\sigma^{-1})x_t - \sigma^{-1}hx_{t-1}] + \epsilon_t^\pi \tag{5}$$

$$\hat{\imath}_t = \phi_r\hat{\imath}_{t-1} + (1-\phi_r)(\alpha_{s_t}\hat{\pi}_{t-1} + \phi_x x_{t-1}) \tag{6}$$

together with the shocks dynamics

$$\epsilon_t^x = \rho_x\epsilon_{t-1}^x + \nu_t^x, \quad \epsilon_t^\pi = \rho_\pi\epsilon_{t-1}^\pi + \nu_t^\pi.$$

In the first equation, the coefficient h corresponds to the degree of (external) habits and σ is the inverse of the relative risk aversion coefficient. In the second equation, the coefficient γ relates to the degree of

[10]We do not work out the determinacy conditions, analog to those derived in section 3, that correspond to this augmented framework. They could be derived along the lines of Bhattarai, Lee, and Park (2014), but the expressions would be cumbersome and come at the cost of a less straightforward intuition.

indexation of prices to past inflation and ω is the inverse of the Frisch elasticity of labor supply. The slope of the Phillips curve, κ, is related to the Calvo probability of not resetting prices ξ according to

$$\kappa = \frac{(1-\beta\xi)(1-\xi)}{\xi(1+\omega\theta)},$$

where θ is the price elasticity of demand. The shocks ν^{π} and ν^{x} are independent and identically distributed, with respective standard deviations σ_{π} and σ_{x}.

The third equation, the policy rule, is the same as in the simple model above, with a notable exception. To facilitate the implementation of the Barthélemy and Marx (2017) approach, we assume that the interest rate in the Taylor-rule reacts to the lagged rather than the current inflation rate. In the solution/simulation, we maintain the smooth transition for policy rule.

The specification for the transition probability from regime s to regime 0 is

$$p_{s,0}(\hat{\pi}_{t-1}) = \begin{cases} 1 & \text{if} \quad |\hat{\pi}_{t-1}| \leq \delta \\ 1 - \exp\left(-\frac{\lambda_p}{|\hat{\pi}_{t-1}|^2 - \delta^2}\right) & \text{otherwise,} \end{cases} \tag{7}$$

Under this specification, if last period's inflation lied within the tolerance band, i.e., $|\hat{\pi}_{t-1}| \leq \delta$, the probability of transiting from regime s to regime 0 is 1. Conversely, if last period's inflation lay outside the tolerance range, the probability of transiting from regime s to regime 0 is a decreasing function of $|\hat{\pi}_{t-1}|$. The parameter $\lambda_p \geq 0$ governs the speed at which this probability decreases (the higher λ_p, the lower the decline).[11]

We take a model period to be a quarter. Parameters are calibrated relying on standard values in the literature, and with a focus on the euro area. We calibrate β to 0.9974, implying a steady-state annualized real interest rate of 1.2 percent. This value corresponds to the average real interest rate in the euro area over the sample 1997Q1-2014Q4 that we use to calibrate shock properties. We also set $\sigma = 1$, thus assuming logarithmic utility derived from consumption, $\omega = 2$ as is customary in the literature, and $\theta = 6$, implying a 20 percent markup. The degree of price stickiness ξ is set to 0.66, consistent with available micro data evidence. Given the values assigned to θ and ω, this yields a slope of the New Keynesian Phillips curve consistent with available euro area estimates. We set $\gamma = 0.2$ and $h = 0.5$. Coming to monetary policy, we calibrate ϕ_r to 0.8, a value consistent with previous euro area estimates. We also set $4\phi_x$ to 0.25, reflecting the primary focus of monetary policy on price stability in the euro area.

We set ρ_{π} to zero and calibrate the other shock parameters so as to broadly match the variance of inflation the variance of HP-filtered, logged GDP, the variance of the short-term nominal interest rate, and the autocorrelation of inflation. Inflation is interpreted as the growth rate of the GDP deflator and

[11] See Appendix for a plot of the transition probability as a function of lagged inflation.

Table 1. Parameter values for robustness exercise

Parameter	Baseline	Low.	High
β	0.9974	–	–
γ	0.2000	0.0000	0.6000
ξ	0.6600	0.5000	0.9000
σ	1.0000	–	–
h	0.5000	0.0000	0.8000
ϕ_r	0.8000	0.0000	0.9000
$4 \times \phi_x$	0.2500	–	–
ρ_x	0.7187	0.3000	0.9000
ρ_π	0.0000	0.0000	0.0000
$100 \times \sigma_x$	0.2025	0.1012	0.4050
$100 \times \sigma_\pi$	0.1557	0.0779	0.3114
400δ	1.0000	0.5000	2.0000

we use the Euribor 3-month rate as the relevant policy rate. Our sample of quarterly data covers the 1997Q1-2014Q4 period.[12]

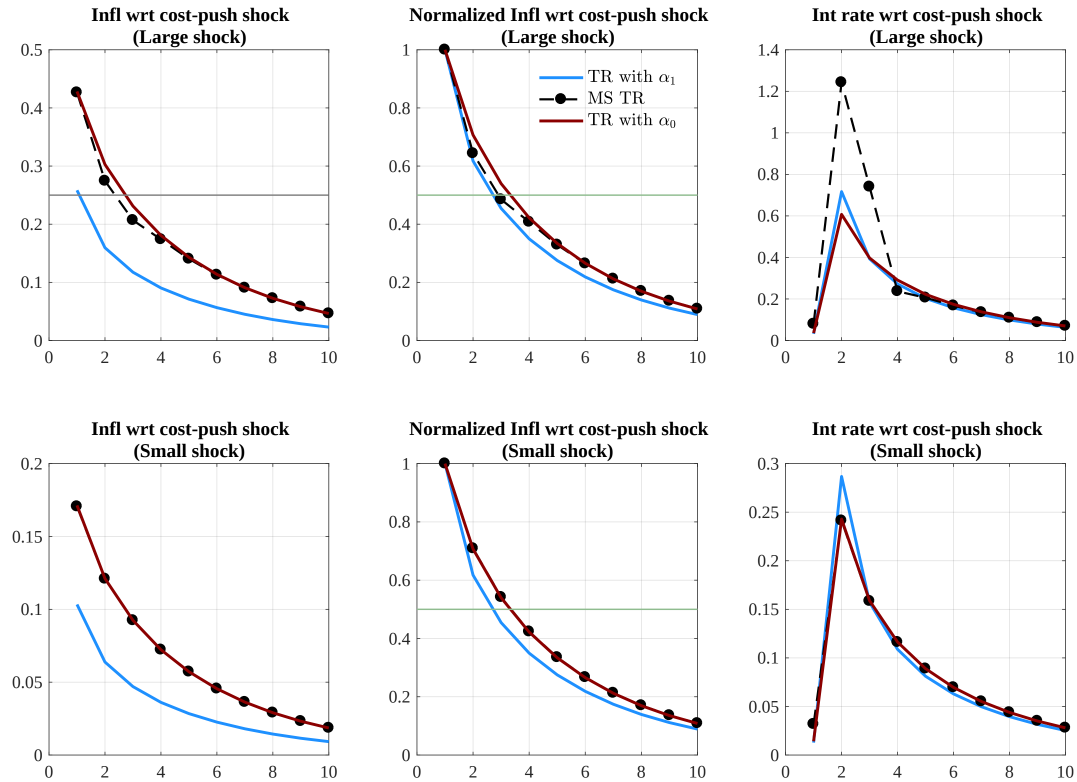
Finally, we calibrate the parameters of the regime-switching policy rule. In particular, the value of δ corresponds to 1 percent in annualized terms, capturing the fact that most central banks that use a range tolerate a deviation of ±1 percentage point from their central target (see Grosse-Steffen 2021). The parameter controlling the transition function is set to $\lambda_p = 6 \times 10^{-8}$. Given the estimated variances of the structural shocks, this value allows us to approximate closely the original process with a threshold (rather than a non-smooth transitions) between regimes. Finally, in the various exercises that follow, we vary α_0 from a minimum value of $\alpha_{\min} = 1.1$ to larger value (up to 3). Parameter α_1 is also varied in the same range (but we mainly focus on policy-relevant cases corresponding to $\Delta\alpha = \alpha_1 - \alpha_0 \geq 0$) .

4.2. **Inflation ranges and policy flexibility.** Using impulse response functions in deviation from the steady state (IRF), we first illustrate how the inflation tolerance range set-up influences interest rate and inflation dynamics after a shock. Given our illustrative purpose here, we use a particular, simplified, calibration of the model and focus on a cost-push shock.[13]

[12]The data are obtained from the ECB-SDW website. In practice, to avoid a stochastic singularity, we also append a monetary policy shock ν_t^r to the Taylor rule. However, in our simulations, this shock is discarded.

[13]In particular, we set the Taylor coefficient in the "tolerant" regime, α_0, to the conventional value of 1.5, while $\alpha_1 = 3$ is the Taylor coefficient when inflation is outside the range. We also set $\gamma = 0$, $\phi_r = 0$, and the cost-push shock has persistence $\rho_\pi = 0.8$

Figure 2. Impulse Response Functions to a Cost-Push Shock

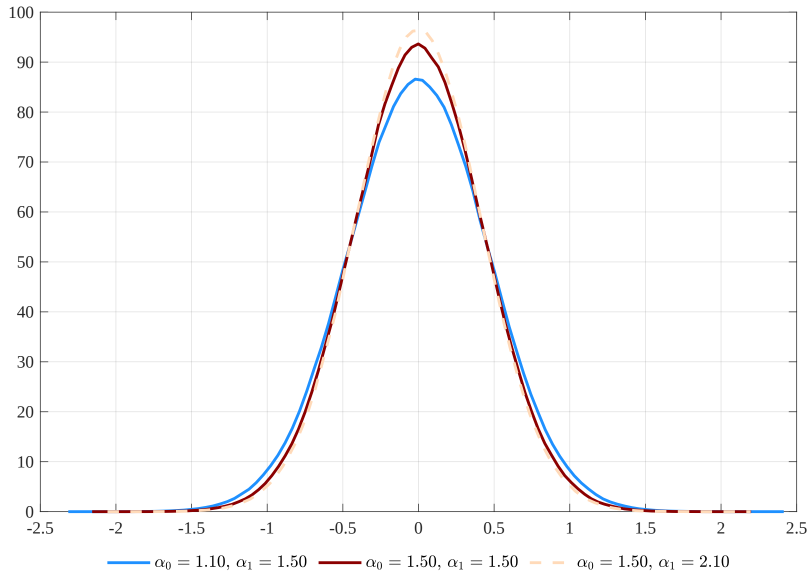


Note: TR stands for Taylor rule, MS stands for Markov-Switching. The dark lines are obtained by averaging 100 IRFs, each corresponding to a particular sequence of regimes s_t. The horizontal line in the top-left panel indicates the upper boundary of the inflation tolerance range, δ. The horizontal line in the top and bottom middle panels serve to identify the half-life of the inflation response.

Results are reported in Figure 2. The top row shows the IRF of inflation, normalized inflation, and the nominal interest rate after a cost-push shock large enough for inflation to leave the tolerance range. The blue curves corresponds to the IRF that would obtain under a Taylor rule with a uniform inflation coefficient set to α_1. By contrast, the red curves corresponds to the IRF obtained under a uniform Taylor rule with α_0. The dark curves with circles correspond to the IRF obtained under the Markov-Switching Taylor rule. The latter is obtained by averaging 100 IRFs, each obtained under a random sequence of regimes s_t. In the top-left panel, the horizontal gray line indicates the upper boundary of the inflation tolerance range, δ. In the middle panel, we show normalized inflation, defined as inflation normalized by its initial response. This allows to identify easily the half-life of inflation's IRF, i.e. the time it takes for inflation to cross the dark green horizontal line set at 0.5. The bottom row reports analog objects, this time obtained after a shock small enough that inflation does not leave the tolerance range.

Under a Markov-Switching Taylor rule, following a cost-push shock large enough for inflation to leave the tolerance range, the dynamics of inflation differ (in a non-proportional way) from the response following a small shock. As a result, the half-life of inflation after a shock that creates a small deviation from the target is slightly longer than that obtained after a shock that pushes inflation far from the target

FIGURE 3. The steady-state distribution of inflation



Note: Each curve shows the distribution of inflation for a given set of policy parameters (α_0, α_1). These distributions are obtained from a model simulation of size 1,000,000 (after discarding a burn-in sample of size 200).

(about four quarters versus three quarters). This illustrates the feature of patience or flexibility attached to the notion of tolerance range. In this setup, the central bank is ready to look through small shocks and let inflation adjust at a gradual pace. Hence, the reaction of the nominal interest rate is much less pronounced under a small shock than with a larger shock, as evidenced in the right-most panels.

Interestingly, after a large shock, the nominal interest rate increases by a larger amount than under the aggressive Taylor rule with an inflation coefficient set to α_1. This is a consequence of agents' forward-lookingness, which Davig and Leeper (2008) refer to as the expectations formation effect. After a large shock, agents witness a strong reaction of the central bank but expect that this reaction will be smaller as soon as inflation gets back into the tolerance range. As a consequence, it takes an even stronger reaction of the nominal interest rate to curb inflation expectations.

4.3. **The distribution of inflation.** Figure 3 shows the distribution of inflation $\hat{\pi}_t$ under different scenarios for the coefficients governing the reaction of the policy rate r_t to inflation π_t in each of the two regimes, (α_0, α_1).

These distributions are obtained by simulating the model over 1,000,000 periods (after having discarded an initial burn-in sample of 200 periods). The blue curve corresponds to our baseline calibration with $\alpha_0 = 1.1$ and $\alpha_1 = 1.5$. Under this configuration, the central bank reacts mildly to inflation in regime 0 (inside the band) and has a conventional degree of reaction to inflation in regime 1 (outside the band). By contrast, the red curve corresponds to a situation in which the central bank has the same conventional reaction inside and outside the band, with $\alpha_0 = \alpha_1 = 1.5$. Finally, the dashed pink curve corresponds to a situation in which the central bank has a conventional reaction to inflation inside the tolerance range and an agressive reaction outside the band ($\alpha_1 = 2.1$). Importantly, the three curves are obtained under the same draw of structural shocks.

The difference between the blue curve and the other two curves illustrates how the role of the reaction inside the band impinges on the volatility of inflation while the reaction outside the band seems to have less traction.[14] In particular, the red curve and the dashed pink curve hardly differ, suggesting that the difference between $\alpha_1 = 1.5$ and $\alpha_1 = 2.1$ does not play a crucial role. In Appendix B, we use in a simplified model to provide some analytical insight on how the reaction outside the band affects the overall inflation volatility.

We now investigate more systematically how the relative size of the reaction when inflation is outside the band affects the macroeconomic outcomes. For this purpose, we conduct the following exercise. We pick a set of values of $\alpha_0 \in [1.1, 1.8]$. In each case, we set $\alpha_1 = \alpha_0 + \Delta\alpha$, with $\Delta\alpha$ ranging from 0 to 1 over a discrete grid of values. We then simulate the model under each possible (α_0, α_1) configuration, using the same draw of structural shocks. To summarize the distribution properties of inflation in these simulations, we focus on the standard deviation of inflation and the percentage of time that inflation spends within the band. We also illustrate how such variations in α_1 impinge on the variance of the nominal interest rate $\hat{\imath}_t$.

The outcome of these simulations is reported in Figure 4. The upper panel shows the standard error of inflation $\hat{\pi}_t$ as a function of $\Delta\alpha$. The darker blue curves correspond to the scenarios with relatively low α_0 while the lighter blue curves correspond to higher values of α_0. The middle panel shows the standard error of the nominal interest rate $\hat{\imath}_t$. The bottom panel shows the percentage of times when inflation lies within the band.

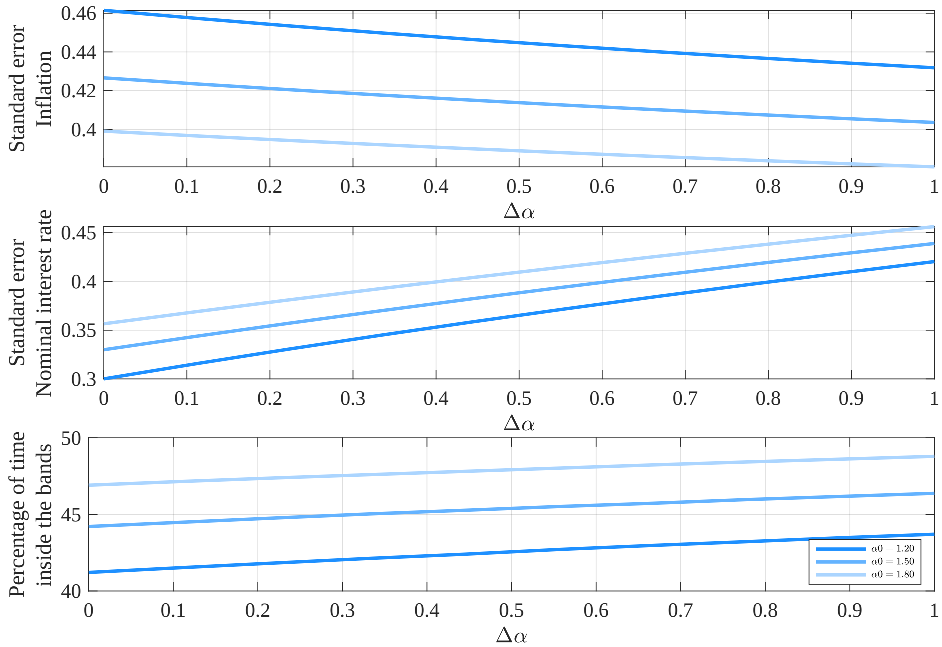
As Figure 4 makes clear, for a given reaction within the band (a given α_0), the standard deviation of inflation decreases as the policy reaction to inflation outside the band increases. This result if fairly intuitive: as the average policy reaction to inflation increases, the volatility of inflation decreases. The flip-side of this coin is that the volatility of the nominal interest increases. Interestingly, even if α_0 is frozen, the unconditional probability of inflation staying within the band increases as $\Delta\alpha$ increases. This illustrates that the policy behavior when inflation is outside the tolerance band is not irrelevant.

4.4. **The trade-off between activism inside vs outside the band.** In this section, we investigate the trade-off between the policy reaction within the band and the reaction outside the band in terms of inflation stabilization.

To this end, we first compute "iso-variance curves", i.e. curves showing the set of policy parameters (α_0, α_1) that yields the same variance of inflation $\hat{\pi}_t$. As before, we simulate the model over 1,000,000

[14]For these simulations, the average probability of inflation being inside the band is around 45%.

FIGURE 4. Moments of inflation as a function on interest rule parameters



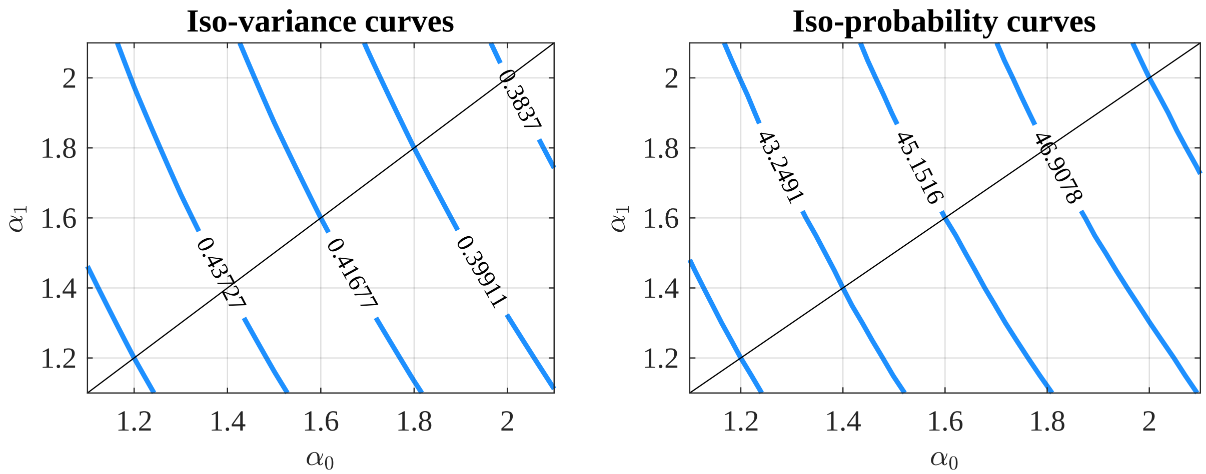
Note: The figure shows the standard error of inflation $\hat{\pi}_t$ (top panel), the standard error of the nominal interest rate $\hat{\imath}_t$ (middle panel), and the probability that inflation stays within the tolerance range (bottom panel) as a function of $\Delta\alpha$ for alternative values of α_0. The blue curves correspond to different α_0, with darker blue curves associated with low values of α_0. The results are based on model simulations of size 1,000,000 (after discarding a burnin sample of size 200).

periods over a grid of (α_0, α_1) configurations, each time using the same draw of structural shocks. We then construct iso-variance curves by interpolation. The results are reported in the left panel of Figure 5. Likewise, we compute iso-probability curves, i.e. curves showing the set of policy parameters (α_0, α_1) that yields the same probability of staying within the inflation band. The results are reported in the right panel of Figure 5.

As the left panel of Figure 5 makes clear, the iso-variance curves are downward sloping and relatively steep. The slope is close to -2.5. Thus, if the central bank decreases its reaction to inflation inside the tolerance range by 0.1 (say moving the Taylor parameter α_0 from 1.5 to 1.4) it has to increase its reaction *outside the tolerance range* by about 0.25 (say moving the Taylor parameter α_1 from 1.5 to 1.75). Put another way, if the central bank seeks to maintain a constant level of inflation variance, it takes a large extra reaction to inflation outside the band to make up for a minor reduction in the reaction to inflation within the band.

Stated in terms of iso-probabilities, the trade-off between the reaction within and the reaction outside the band also appears unfavorable. As the right panel of Figure 5 shows, the iso-probability curves are also downward sloping, with approximately the same slope as iso-variance curves. Thus, if the central bank decreases its reaction to inflation inside the tolerance range by 0.1 (again, moving the Taylor parameter

FIGURE 5. Iso-variance and iso-probability curves



Note: The left panel shows the iso-variance curves. Each blue line gives the combination of parameters α_0 and α_1 yielding the same variance of inflation. For ease of interpretation, the figure also reports the 45 degree line. The right panel shows iso-probability curves. Each blue line gives the combination of parameters α_0 and α_1 yielding the same probability that inflation stays inside the tolerance range.

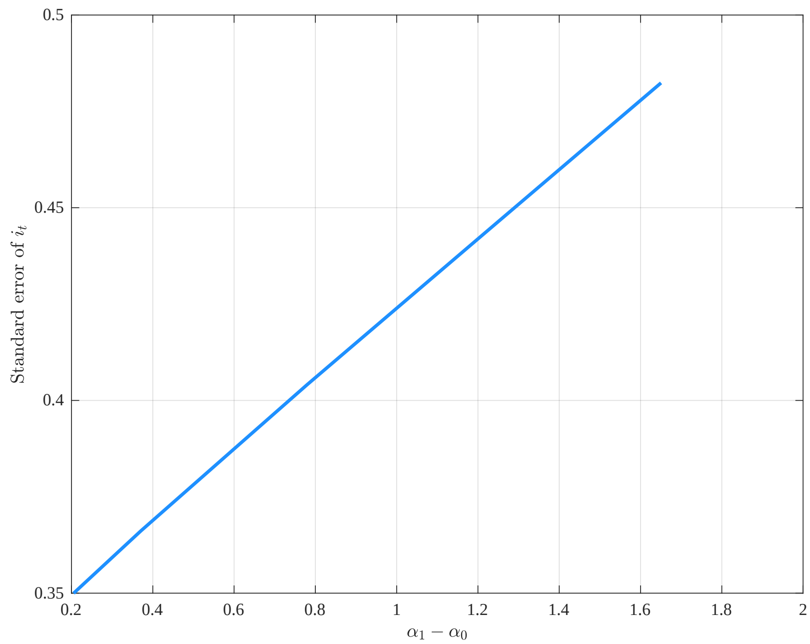
α_0 from 1.5 to 1.4) it has to increase its reaction *outside the tolerance range* by about 0.25 (moving the Taylor parameter α_1 from 1.5 to 1.75) to maintain the probability of staying within the tolerance range constant.

One reason why the iso-variance is steep (and the trade-off unfavorable) is that the "agressive" region is not visited frequently, so the more active policy parameters gets less traction. In addition, whenever a given shock pushes inflation outside the tolerance range (say above the upper boundary), forward-looking agents expect that at some point in the future, with a positive probability, policy will revert to its less aggressive mode. As a consequence, the convergence speed of inflation to steady state will be reduced. This holds even immediately after the shock.

This mechanism can be illustrated by the IRFs reported in Figure 2. There, we show the dynamics following a cost-push shock in a model with bands (dark curves with circles) as compared to a model with either only an aggressive rule (blue curves), or only a tolerant rule (red curves). After a shock of a sufficiently large size (upper panel), inflation is out of the range in the first period. However, in the set-up with a tolerance range, even if the current reaction function is governed by the same parameter as with the aggressive rule, the dynamics of inflation is more persistent than under the aggressive rule. It is actually closer to the trajectory observed when policy is governed by the tolerant rule. Indeed, the forward looking nature of inflation attenuates the benefits of having a very aggressive rule in the "out-of-the-range" region.

To complement on these results, we illustrate another dimension of the trade-off. Figure 6 plots the variance of the nominal interest rate as a function of $\Delta_\alpha = \alpha_1 - \alpha_0$ for the set of pairs of parameters (α_0, α_1) that yield the same variance of inflation as the parameter pair $\alpha_0 = \alpha_1 = 1.5$. Thus, we investigate

FIGURE 6. Interest rate variability



Note: The figure plots the nominal interest rate variance as a function of $\Delta\alpha = \alpha_1 - \alpha_0$ for the set of pairs of parameters (α_0, α_1) that yield the same variance of inflation as the parameter pair $\alpha_0 = \alpha_1 = 1.5$

how the variance of the nominal interest rates varies as one moves along a given iso-variance curve for inflation.

As Figure 6 makes clear (and as was already hinted by Figure 2), this relation is increasing. Thus, increasing the tolerance to inflation when it is close to the target, while at the same time increasing the reaction to inflation when inflation is outside the range, so as to to achieve the same level of inflation variability as in the no range model, results in a significantly larger level of interest rate variability. Arguably interest rate variability is ceteris paribus - i.e. controlling for the level of macroeconomic volatility -an undesirable feature, for instance for financial stability reasons. From this perspective, increasing tolerance within the range, while increasing activism outside the range, results in an unfavorable outcome.

4.5. **Robustness.** To complement on the previous analysis, we investigate the robustness of our results to alternative parameter values. We subject a set of key parameters to perturbations to assess the implied impact on the iso-variance curves and iso-probability curves in the (α_0, α_1) plane.

The parameters being investigated are: the degree of interest rate smoothing ϕ_r, the degree of indexation of prices to past inflation γ, the degree of habit formation in the IS equation h, and the price stickiness parameter ξ. We also investigate the width of the tolerance range δ, the standard error of demand shocks σ_x, the standard error of supply shocks σ_π, and the persistence of demand shocks ρ_x.

For each of the parameters subject to a perturbation, we consider high and low values that fall if anything beyond the available range of estimates. To some extent, this robustness analysis thus can viewed as extreme. For the parameter δ, we consider tolerance ranges that are either shrunk or widen by a factor 2, leading to a tolerated deviation of respectively ± 0.5, or ± 2 annualized percentage point around the central target. For each parameter perturbation, we compute iso-variance and iso-probability curves following the exact same procedure as before.

The alternative values of parameters are presented in Table 1. The iso-variance results are presented in Figure 7. Likewise, the results for the iso-probability curves are reported in Figure 8. In each case, the blue curves correspond to the low parameter value reported in Table 1 while the red curves correspond to the high parameter value.

Let us first consider the case of the shock variances. If any of the latter increases, the probability that inflation leaves the tolerance range increases as well, as is apparent from Figure 8. When a larger fraction of time is spent outside the inflation band, the influence of parameter α_1 is mechanically larger. Given this larger traction of α_1, a relatively mild increase in this parameter suffices to compensate for a smaller parameter α_0 in terms of the overall volatility of inflation. As a result, when the shock variances are large, we obtain somewhat flatter iso-variance curves in Figure 7. This effect remains quantitatively small, albeit more apparent in the case of demand shocks volatility. The iso-probability curves are, for the same reasons, flatter with larger shocks. By the same line of reasoning, when the persistence of demand shocks ρ_x increases, the overall variance of demand shocks increases, so that, once again, the iso-variance and the iso-probability curves become less steep.

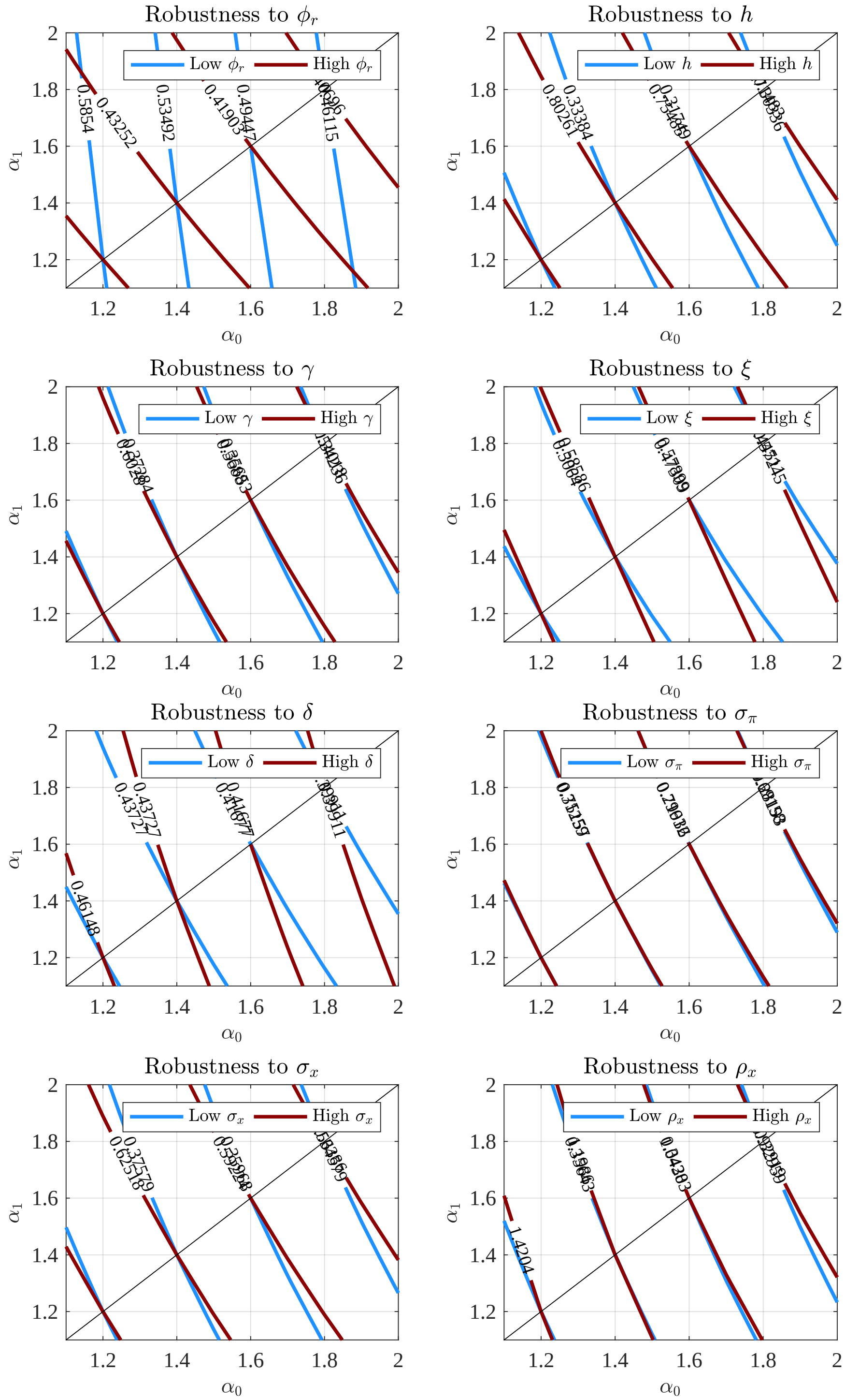
Consider then the parameter δ governing the width of the tolerance range. To some extent, a smaller δ is isomorphic to larger shocks. Obviously, if δ decreases, the probability of inflation leaving the tolerance range increases. Once again, this gives more traction to α_1. Thus, as δ decreases, the slope of the iso-variance and iso-probability curves decreases in absolute value. By contrast for a very wide band the iso variance curves become extremely steep: it is virtually impossible to compensate any decrease in parameter α_0 by augmenting α_1.

We now consider the case when the slope of the Phillips curve steepens, i.e. when the Calvo probability ξ decreases. Inflation then becomes more and more sensitive to demand shocks. Again this increases the probability that it leaves the tolerance range. Here too, this gives more traction to α_1, making iso-variance and iso-probability curves less steep.

Similarly, when the degree of habits h increases, the intrinsic persistence of the output gap x_t increases so that it has a more pronounced effect on inflation (for a given slope of the Phillips curve). Again, this makes the iso-variance and iso-probability curves less steep. The quantitative importance of this effect, though, is moderate. The same logic applies to the intrinsic persistence of inflation γ.

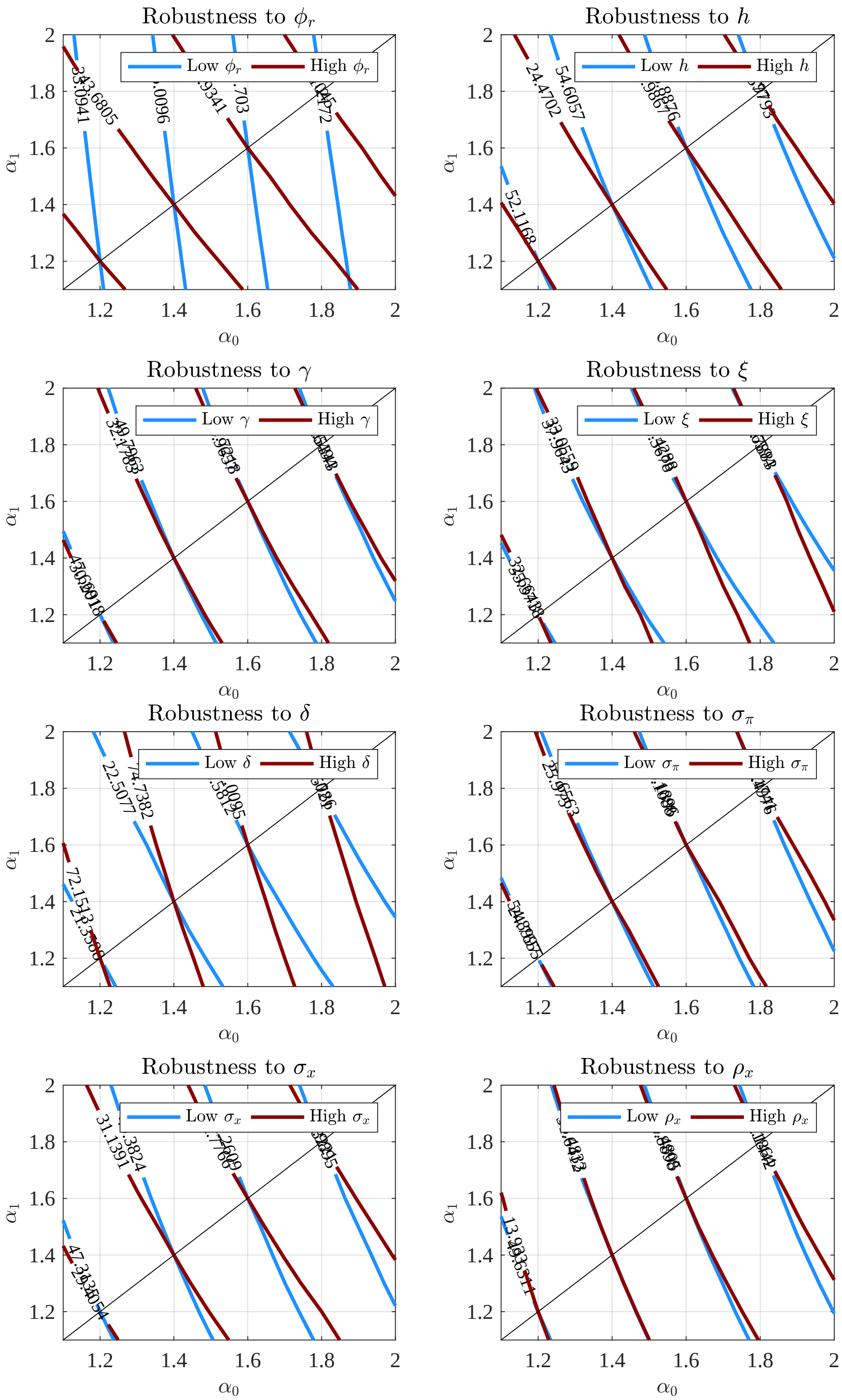
Finally, when the degree of interest rate smoothing ϕ_r increases, the policy rule is less reactive on impact but overall more stabilizing, reflecting the powerful expectation effects of monetary policy. With

FIGURE 7. Iso-variance curves for alternative parameter values



Note: Each sub-figure shows iso-variance curves when a given parameter value is altered. The parameters considered: ϕ_r (the degree of interest rate smoothing), γ (the degree of indexation of prices to past inflation), h (the degree of habit formation in the IS equation), ξ, (the Calvo probability), the standard error of demand shocks σ_x, the standard error of the cost-push shock σ_π, and the persistence of the demand shock ρ_x. For each parameter, we consider perturbations around the benchmark value, with a lower and a larger value, as reported in Table 1.

FIGURE 8. Iso-probability curves for alternative parameter values



Note: Each sub-figure shows iso-probability curves when a given parameter value is altered. The parameters considered: ϕ_r (the degree of interest rate smoothing), γ (the degree of indexation of prices to past inflation), h (the degree of habit formation in the IS equation), ξ, (the Calvo probability), the standard error of demand shocks σ_x, the standard error of the cost-push shock σ_π, and the persistence of the demand shock ρ_x. For each parameter, we consider perturbations around the benchmark value, with a lower and a larger value, as reported in Table 1.

larger ϕ_r inflation spends more time within the tolerance range. However, the overall stabilizing effect of parameter α_1 appears to be strengthened, resulting in less steep iso-variance and iso-probability curves.

We do not report robustness with respect to β, as the sensitivity of iso-variance and iso-probability curves to this parameter turns out to be very limited. In the next section (Section 5), devoted to the case with a ZLB, we will adjust the calibration of this parameter consistently with the targeted value of the steady-state real natural rate $r^\star$. It is thus comforting that the results described here are weakly affected by the calibration of β.

All in all, the main insights obtained in the case of the baseline calibration appear to be very robust to the values of the selected parameters.

5. The case of a relevant Zero Lower Bound

5.1. **Motivation.** So far our analysis has abstracted from issues raised by the zero lower bound (ZLB). Yet, the ZLB (or effective lower bound) on the level of interest rate is empirically relevant in the context of a low "natural rate" of interest environment. The long spell of interest rate at zero or below zero during and after the financial crisis in the US or the euro area (and then again at the time of the recession triggered by the Covid 19 crisis) is an obvious demonstration of this.

Taking the ZLB on the interest rate into account in our theoretical environment has the potential to alter our quantitative assessment on interest rate rule with tolerance ranges. In particular, with a more reactive policy rule outside the band, or with a wider band, it might be the case that the ZLB raises a more frequent concern than under a standard linear policy rule. In this section, we investigate this issue.

5.2. **Steady state with the ZLB.** First, we study how the ZLB leads to existence of a second equilibrium, in a simplified version of model (1)-(3), with $\phi_r = \phi_x = 0$, taking into account the ZLB constraint. This is precisely the same phenomenon as the double equilibria investigated by Benhabib et al. (2001). Under these simplifying assumptions, the model boils down to

$$
\begin{aligned}
x_t &= \mathbb{E}_t\{x_{t+1}\} - \sigma(\hat{\imath}_t - \mathbb{E}_t\{\hat{\pi}_{t+1})\} + \nu_t^x \\
\hat{\pi}_t &= \beta\mathbb{E}_t\{\hat{\pi}_{t+1}\} + \kappa x_t + \nu_t^\pi \\
\hat{\imath}_t^S &= \alpha_{s_t}\hat{\pi}_t \\
\hat{\imath}_t &= \max\{\hat{\imath}_t^S, -\bar{\imath}\}
\end{aligned}
\tag{8}
$$

We have then the following result:

Proposition 3. *If* $\min\{\alpha_0, \alpha_1\} > 1$*, then model* (8) *admits on top of the steady state* $\hat{\pi} = 0$*,* $\hat{\imath} = 0$*, another steady state at the ZLB,* $\hat{\pi}^*_{ZLB} = -\bar{\imath}$*,* $\hat{\imath}^*_{ZLB} = -\bar{\imath}$*. Conversely, if* $\max\{\alpha_0, \alpha_1\} < 1$*, there exists a unique steady state* $\hat{\pi} = 0, \hat{\imath} = 0$*.*

The proof of Proposition 8 relies on simple analytic computations and is relegated in Appendix. The Appendix also illustrates that a model with a kink (rather than a jump) at the edge of the tolerance range, results in actually 3 or 4 equilibria, rather than 2. Furthermore, we notice that, in the first case, when $\min\{\alpha_0, \alpha_1\} > 1$, the equilibrium $\hat{\pi} = 0$, $\hat{\imath} = 0$ is expectationally stable in the sense of Evans (1985), while it is not the case of the ZLB steady state. If $\max\{\alpha_0, \alpha_1\} < 1$, the steady state is not expectationally stable.

5.3. **Simulating the model under ZLB.** We introduce ZLB in the augmented model of section 4, and simulate the model, still resorting to the method designed by Barthélemy and Marx (2017). This strategy relies on local approximation. Our method is in the same vein as Guerrieri and Iacoviello (2015). Here, we consider that the only steady state in the model is $\hat{\pi} = 0$, and $\hat{\imath} = 0$.

To obtain empirically realistic probabilities of hitting the ZLB, we deviate from the benchmark calibration and consider a permanent negative shock on the natural interest rate, resulting in more frequent ZLB episodes (see Table 3).

In this setup, we have to consider formally four regimes: (i) inflation is within the band and the ZLB does not bind, (ii) inflation is within the band and the ZLB binds, (iii) inflation is outside the band and the ZLB does not bind, and (iv) inflation is outside the band and the ZLB binds. The four regimes are summarized in Table 2.

Regime (ii) might appear an implausible regime, in particular in view of Figure (1). However, recall that in the extended model, there is an output gap term in the reaction function, so that there is a possibility that interest rate is driven to or kept at the ZLB due to the output gap movements, or to interest rate inertia, while inflation is actually in the tolerance range. In addition, whenever the tolerance range is wide (ie for large values of δ) the interest rate might be driven to its lower bound without inflation leaving the tolerance range.

To carry out the simulations in this section, the inflation target is set to 2 percent, and the steady-state value of the real interest rates set to $r^\star = 0.5$ percent. This specification corresponds to a conservative calibration of the "New Normal" environment with a relatively low steady-state real natural interest rate. As before, moments under the various parameter sets are obtained by simulating the model over 1,000,000 periods (after having discarded an initial burn-in sample of 200 periods).

5.4. **Result under the baseline, and under larger shocks.** To begin with, we explore the consequences of taking the ZLB into account on the distribution properties of inflation. Table 3 illustrates the impact of taking the ZLB into account on average inflation and the standard deviation of inflation, and on the frequency of ZLB episodes. In the baseline calibration, the outcome of which is reported on the line labelled "ZLB", the impact of the ZLB appears very small. Indeed, the probability of hitting the ZLB is a mere 4.3%, resulting in only minor differences compared to the no-ZLB case. We thus consider an

Table 2. The four regimes

	Inflation	
	Inside the band	Outside the band
Interest rate above ZLB	$s_t = 1$ $\lvert\pi_{t-1} - \pi^*\rvert < \delta$ $\hat{\imath}_t > -\bar{\imath}$	$s_t = 3$ $\lvert\pi_{t-1} - \pi^*\rvert > \delta$ $\hat{\imath}_t > -\bar{\imath}$
at ZLB	$s_t = 2$ $\lvert\pi_{t-1} - \pi^*\rvert < \delta$ $\hat{\imath}_t = -\bar{\imath}$	$s_t = 4$ $\lvert\pi_{t-1} - \pi^*\rvert > \delta$ $\hat{\imath}_t = -\bar{\imath}$

Note: $\hat{\imath}_t$ represents the Taylor rate $\phi_r \hat{\imath}_{t-1} + (1 - \phi_r)(\alpha_{s_t} \hat{\pi}_{t-1} + \phi_x x_{t-1})$

Table 3. Moments under ZLB

	P(ZLB)	Mean$(\pi_t - \pi^\star)$	s.d.$(\pi_t - \pi^\star)$
No ZLB	–	-0.00	1.25
ZLB	4.3	-0.01	1.26
ZLB (large shocks)	17.5	-0.17	2.32

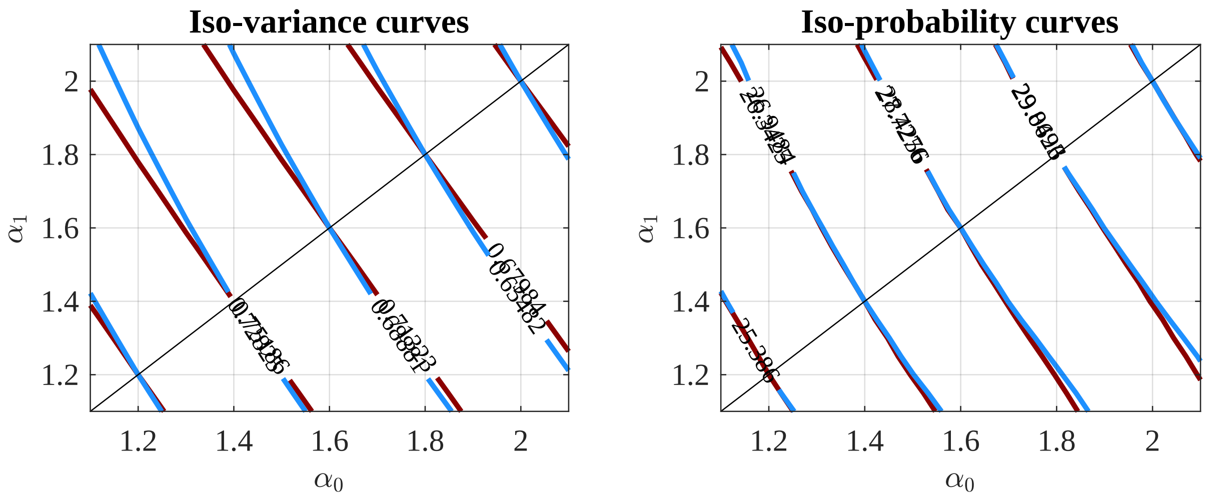
Note: P(ZLB) in percent. Mean and standard deviation are in percent, annualized. The reactions inside and outside the band are $\alpha_0 = \alpha_1 = 1.5$.

alternative calibration in which the ZLB is a more significant threat. To this aim, we recalibrate the demand shocks[15], allowing for larger recessions resembling the global financial crisis. The line labelled “ZLB (larger shocks)” in Table 3 shows that under our alternative calibration of demand shocks, the probability of hitting the ZLB is substantially higher, at 17.5%. This value is more line with the existing assessments of the unconditional probability of ZLB episodes in the “New Normal” environment. (Table 4 in Appendix shows the different probabilities of hitting the ZLB for different configurations of (α_0, α_1) and different steady-state interest rates in the case of larger shocks, further suggesting those are a relevant benchmark.) In the setup, the negative inflation bias induced by the ZLB is about -17 basis points. Likewise, the standard error of inflation is now nearly twice as high as absent the ZLB.

To investigate how taking the ZLB into account affects the trade-off induced by inflation tolerance band, we compute, as before, iso-variance curves, i.e. curves showing the set of policy parameters (α_0, α_1) that yield the same variance of inflation $\hat{\pi}_t$. Again, we simulate the model over 1,000,000 periods for

[15]The standard error $\sigma_x = 0.2$ in the baseline calibration is increased to $\sigma_x = 0.5$ in the case of large shocks, all the other parameters remain unchanged.

Figure 9. Iso-variance and iso-probability curves : baseline vs ZLB case



Note: The coordinates of each point on a given curve correspond to the combination of parameters (α_0, α_1) yielding the same variance of the inflation rate, allowing for a ZLB constraint in the simulations. Blue lines correspond to the no-ZLB case and red lines correspond to the ZLB case.

different (α_0, α_1) configurations, each time using the same draw of structural shocks. We then construct iso-variance curves by interpolation. Results are reported in the left panel of Figure 9. Likewise, we compute iso-probability curves, i.e. curves showing the set of policy parameters (α_0, α_1) that yield the same probability of staying within the inflation band. Results are reported in the right panel of Figure 9.

Figure 9 shows that the variance of inflation is larger with than without ZLB, as hinted at by Table 3. This is a direct consequence of the skewed distribution of inflation obtained when the ZLB is taken into account. However, the the trade-off between activism inside versus outside the band introduced by the tolerance range is still present under our calibration. Moreover, the iso-variance curves resemble their no-ZLB counterparts. Overall results obtained in the previous section on the trade-off between activism inside and outside the tolerance range, carry out nearly identically to the case allowing of a relevant ZLB constaint.

6. Conclusion

We have studied the properties of inflation ranges policies in a standard, New Keynesian set-up. Both through establishing some analytical results, and through simulations of a quantitative, empirically relevant version of the New Keynesian model, we have established that relying on inflation ranges raise important concerns in terms of macroeconomic stabilization. Our analysis has provided a formal basis for the recognition by policymakers that inflation tolerance ranges should not be interpreted nor implemented as inaction range.[16] Further, we have shown that a central bank embarking in a tolerance range policy

[16] An example of such a claim is provided by Coeuré (2019): "Such a tolerance band, which can be more or less precise, is not an invitation for inaction or complacency."

should be ready to react very strongly to inflation deviation whenever inflation falls outside the tolerance range, to compensate for lower reaction within the range - entailing added interest rate volatility. Finally, we have also shown that these results are robust, as they broadly hold for a wide constellation of parameters of the NK model, or when allowing for the zero (or effective) lower bound to be a relevant concern.

An obvious limitation of our model it that we have not made explicit the potential benefits of inflation tolerance bands, and thus do not provide a full assessment of these policies. To further explore the trade-off involved with inflation ranges, at least two directions, each raising additional technical challenges, could be envisioned. First, an issue is whether inflation inaction (or limited action) ranges can be derived as an optimal policy, when incorporating either a non-linear Phillips curve, or a non-quadratic loss function for the central bank (e.g., allowing for a flat section corresponding to the tolerance range), following Orphanides and Wieland (2000). While Orphanides and Wieland (2000) have investigated such foundations for inaction ranges in a backward looking model, it remains to be established whether similar results hold in a forward looking-model, and to what extent the concerns raised in our paper still apply in such an extended set-up. Second, an alternative route to deriving inflation range as a meaningful policy setup would require other ingredients such as communication with the public, credibility gain and losses of the central bank, or monitoring the central bank. While this question is interesting per se and would capture the essence of "uncertainty ranges", it presupposes a meaningful departure from full-information rational expectations – a standard assumption in the baseline NK model. Both these avenues are left for future research.

Appendix A. Proof of Proposition 2

Let us consider the following model

$$\mathbb{E}_t f(X_{t+1}, X_t, X_{t-1}, \sigma\epsilon_t, s_t) = 0 \tag{9}$$

We assume that

H1: Shocks are bounded.

H2: The model admits a steady state, independent of regimes.

H3: The steady-state probability to stay in $s_t = 0$ is 1.

H4: The transition probabilities are twice continuously differentiable, and the steady-state first derivative of the probability to stay in regime 0 is zero.

Proposition 4. *Under assumptions H1-H4, the following results hold:*

(1) *If the linear model obtained by linearizing model (9) around the steady state is not determinate, then there exist stationary sunspot equilibria arbitrarily close to the steady state.*

(2) *Reciprocally, if the linearized model is determinate, for shocks small enough, there exists a unique solution to model (9).*

Proof. The first point is a direct consequence of Theorem 1 in Woodford (1986). The second point is an application of Proposition 1 in Barthélemy and Marx (2017).

■

Proposition 4 shows that determinacy of the models with tolerance range completely relies on the determinacy of the model in regime 0. The determinacy of this model is a straightforward extension of Proposition 4.4 in Woodford (2011) to take into account persistence of the shocks. The model can be written as

$$\mathbb{E}_t z_{t+1} = \tilde{A} z_t$$

with

$$z_t = \begin{bmatrix} \hat{\pi}_t & x_t & \hat{\imath}_{t-1} & \epsilon^{\pi}_{t-1} & \epsilon^{x}_{t-1} \end{bmatrix}', \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad B = \begin{bmatrix} \rho_\pi & 0 \\ 0 & \rho_x \end{bmatrix}$$

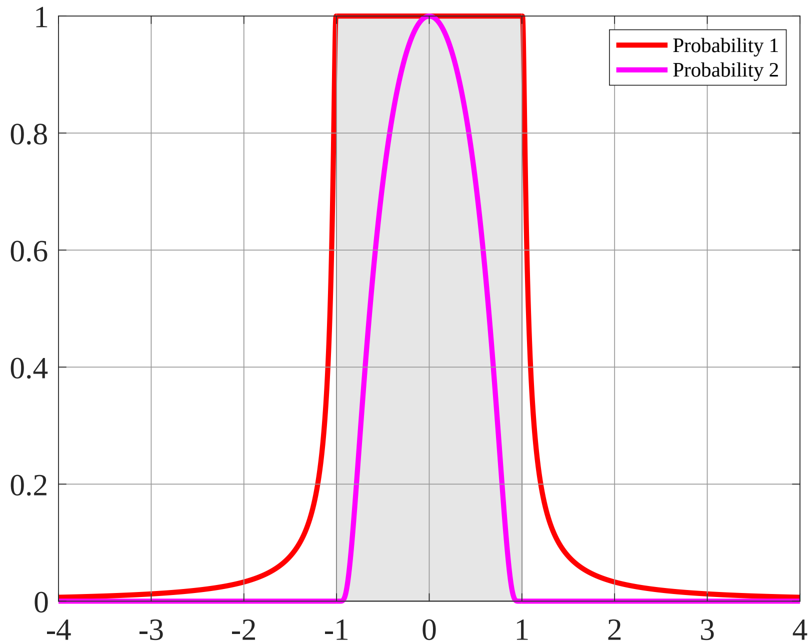
and A is defined in (C.21) in Woodford (2011) .

Noticing that the number of eigenvalues of $\tilde{A}$ larger than 1 is the same as the number of eigenvalues of A larger than 1, we deduce the determinacy condition.

Examples of probabilities consistent with Proposition 4 are represented in Figure 10, and depicted by equations (7) and (10).

$$\tilde{p}_{s,0}(\hat{\pi}_{t-1}) = \begin{cases} \exp\left(\lambda_s\left(1 - \frac{\delta^2}{\delta^2 - |\hat{\pi}_{t-1}|^2}\right)\right) & \text{if} \quad |\hat{\pi}_{t-1}| \leq \delta \\ 0 & \text{if} \quad |\hat{\pi}_{t-1}| \geq \delta \end{cases} \tag{10}$$

FIGURE 10. Probability to stay in regime 0



Note: The figure plots different possible specifications for probability to switch to regime 0, depending on lagged inflation gap, probability 1 is described in (7), while probability 2 is described in (10).

APPENDIX B. IMPACT OF $\Delta\alpha$ ON THE VARIANCE OF INFLATION: ANALYTICAL RESULTS IN A SIMPLIFIED SET-UP

In this section, we provide some analytical insights on how the reaction outside the band may impact the volatility of inflation, in a simplified setup.

We consider the model that collapses to a single equation

$$\pi_t = \alpha_{s_t}\mathbb{E}_t\pi_{t+1} - \sigma r_t$$

with

$$s_t = 0 \qquad \text{if} \quad |\hat{\pi}_{t-1}| < 1$$

$$s_t = 1 \qquad \text{if} \quad |\hat{\pi}_{t-1}| > 1$$

If $\min(\alpha_0, \alpha_1) > 1$, there exists a unique solution

$$\pi_t = \sigma\frac{r_t}{\alpha_{s_t}}$$

Straightforward calculations yield

$$\mathrm{var}(\pi) = p(s_t = 0) \times \frac{\sigma^2 \mathrm{var}(r)}{\alpha_0^2} + p(s_t = 1) \times \frac{\sigma^2 \mathrm{var}(r)}{\alpha_1^2}$$

$$\mathrm{var}(\pi) = \frac{\sigma^2 \mathrm{var}(r)}{\alpha_0^2} \times \left[1 - p(s_t = 1) \times \left(1 - \frac{\alpha_0^2}{\alpha_1^2}\right)\right]$$

Notice that

$$1 - \frac{\alpha_0^2}{\alpha_1^2} = 2\frac{\Delta\alpha}{\alpha_0}$$

and that

$$p(s_t = 1) \leq \int_{|r| > \frac{\sigma}{\max(\alpha_0, \alpha_1)}} dr$$

Now, assuming that r_t follows a Gaussian distribution, we get, from the properties of truncated gaussian distributions:

$$\mathrm{std}(\pi) = \mathrm{std}_0(\pi) \left(1 - \Delta\alpha O(e^{-\max(\alpha_0, \alpha_1)/\sigma})\right)$$

where std_0 represents the standard error of inflation in the model with a linear Taylor rule with a single Taylor parameter α_0, $\mathrm{std}_0(\pi) = \frac{\sigma \mathrm{std}(r)}{\alpha_0}$. It is clear from the above formula that inflation variance is a decreasing function of the "reaction spread" $\Delta\alpha$.

Appendix C. Proof of Proposition 3

Proposition 3 is a corollary of the following lemma:

Lemma 5. *We consider a real number $\bar{\imath} > 0$, and a function a such that $\inf_{\hat{\pi}} a(\hat{\pi}) > 1$, then the equation*

$$\hat{\pi} = \max\{a(\hat{\pi})\hat{\pi}, -\bar{\imath}\}$$

admits exactly two solutions.
Reciprocally, if $\sup_{\hat{\pi}} a(\hat{\pi}) < 1$, then 0 is the only solution of this equation.

There are at most two solutions $\hat{\pi} = -\bar{\imath}$, and $\hat{\pi} = 0$. Indeed, if $a(\hat{\pi})\hat{\pi} = \hat{\pi}$, $(a(\hat{\pi}) - 1)\hat{\pi} = 0$. We just have to check that $\max\{-a(-\bar{\imath})\bar{\imath}, -\bar{\imath}\} = -\bar{\imath}$, which is obvious since $a(-\bar{\imath})\bar{\imath} < -\bar{\imath}$. The problem is represented in Figure 11.

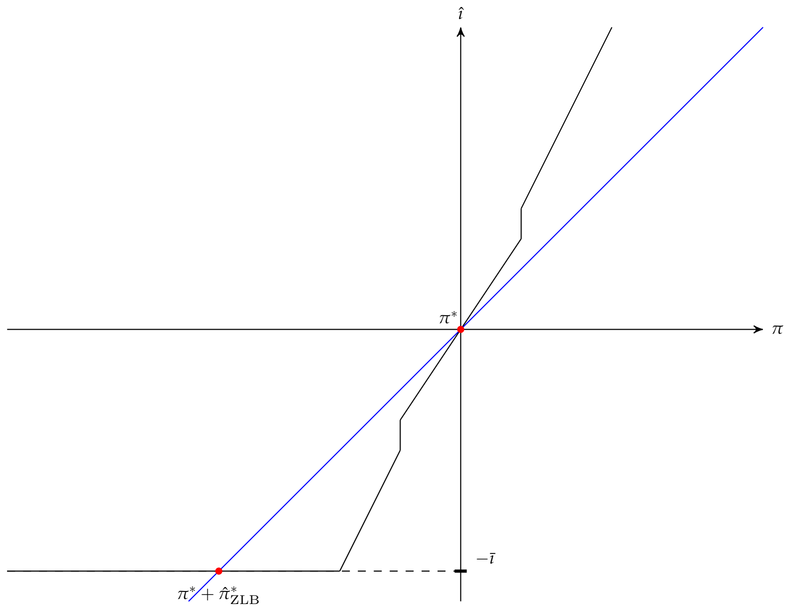
Concerning expectational stability of these steady states, it suffices to compute the spectral radius of A in the model

$$\begin{bmatrix} x_t \\ \hat{\pi}_t \end{bmatrix} = A\mathbb{E}_t \begin{bmatrix} x_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix}$$

where the matrix A satisfies

$$A = \left(\begin{bmatrix} 1 & \sigma\alpha_0 \\ -\kappa & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \sigma \\ 0 & \beta \end{bmatrix}$$

FIGURE 11. Two steady states



We know that it is smaller than 1 if and only if $\alpha_0 > 1$. Moreover, near the ZLB steady state,

$$A = \left(\begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \sigma \\ 0 & \beta \end{bmatrix}$$

This shows that if $\min\{\alpha_0, \alpha_1\} > 1$, the steady-state is expectationally stable, while it is not if $\max\{\alpha_0, \alpha_1\} < 1$. Moreover, the expectational stability around the ZLB steady-state corresponds to $\alpha_0 = 0$, and is thus expectationally unstable.

APPENDIX D. CASE OF A KINKED RULE

The steady-state implications of the kinked Taylor rule is illustrated in Figure 12. The following two lemmas prove our claim that under such a configuration, there are 3 or 4 steady-state equilibria.

Lemma 6. *Fix $\delta > 0$ and (α_0, α_1) two positive real numbers. The solutions of equation*

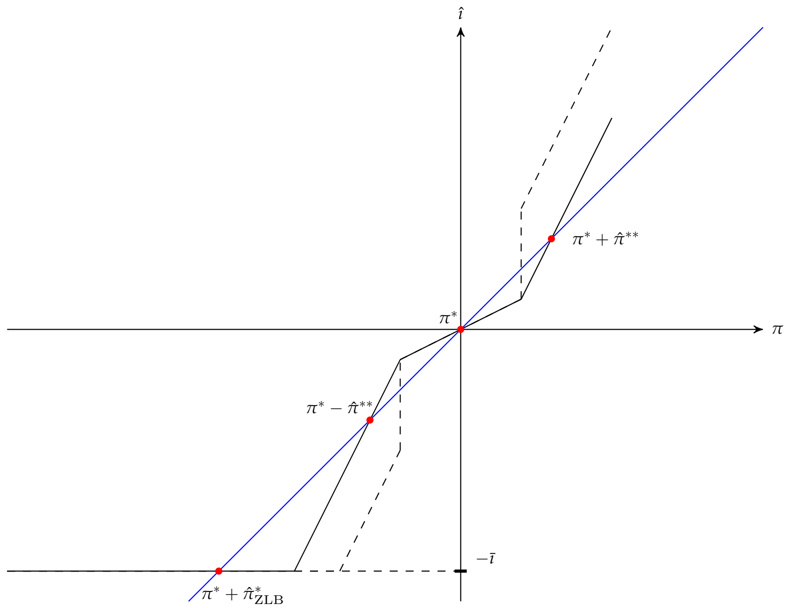
$$\hat{\pi} = \alpha_0 \hat{\pi} \mathbb{1}_{|\hat{\pi}|<\delta} + [\alpha_0 \delta + \alpha_1(\hat{\pi} - \delta)] \times \mathbb{1}_{\hat{\pi}>\delta} + [-\alpha_0 \delta + \alpha_1(\hat{\pi} + \delta)] \times \mathbb{1}_{\hat{\pi}<-\delta} \tag{11}$$

satisfy

- *if* $\max\{\alpha_0, \alpha_1\} < 1$*, or* $\min\{\alpha_0, \alpha_1\} > 1$*,* $\hat{\pi} = 0$ *is the unique solution to equation* (11)*.*
- *if* $\min\{\alpha_0, \alpha_1\} < 1$*, and* $\max\{\alpha_0, \alpha_1\} > 1$*, there exist three solutions to equation* (11)*.*

Proof. We notice that $\hat{\pi} = 0$ is solution, and that the equation is even.
If $\min\{\alpha_0, \alpha_1\} < 1$, and $\max\{\alpha_0, \alpha_1\} > 1$, the function $(\alpha_1 - \alpha_0)\frac{a(\hat{\pi})}{\hat{\pi}}$ is strictly increasing on $[\delta, +\infty[$, and thus $\hat{\pi} \mapsto \frac{a(\hat{\pi})}{\hat{\pi}}$ is a bijection.

Figure 12. Multiple equilibria with a kinked Taylor rule



The bounds satisfy $\frac{a(\hat{\delta})}{\hat{\delta}} = \alpha_0$ and $\lim_{\hat{\pi}\to+\infty} \frac{a(\hat{\pi})}{\hat{\pi}} = \alpha_1$, so there exists a unique $\hat{\pi}^{**}$ in $[\delta, +\infty[$ solution of equation (11).

Since $\alpha_0 \neq 1$, there is no solution on $]0, \delta]$. The solutions are thus $0, -\hat{\pi}^{**}$ and $\hat{\pi}^{**}$.

If $\max\{\alpha_0, \alpha_1\} < 1$, or $\min\{\alpha_0, \alpha_1\} > 1$, the image by the function $\hat{\pi} \mapsto \frac{a(\hat{\pi})}{\hat{\pi}}$ of $]\delta, +\infty[$ is $]\min\{\alpha_0, \alpha_1\}, \max\{\alpha_0, \alpha_1\}[$ and does not contain 1. Since $\alpha_0 \neq 1$, there is no solution on $]0, \delta]$. Thus, the only solution of equation (11) is 0. ■

Lemma 7 (ZLB & kinked Taylor rule)**.** *Fix $\delta > 0$, $\alpha_0 < 1 < \alpha_1$ two positive real numbers, and $\bar{\imath} > 0$. The solutions of equation*

$$\hat{\pi} = \max\{-\bar{\imath}, \alpha_0 \hat{\pi} \mathbb{1}_{|\hat{\pi}|<\delta} + [\alpha_0\delta + \alpha_1(\hat{\pi} - \delta)] \times \mathbb{1}_{\hat{\pi}>\delta} + [-\alpha_0\delta + \alpha_1(\hat{\pi} + \delta)] \times \mathbb{1}_{\hat{\pi}<-\delta}\} \tag{12}$$

satisfy

- *if $\frac{\delta(\alpha_1-\alpha_0)}{\alpha_1-1} < \bar{\imath}$, there exists four multiple equilibria, $-\bar{\imath}$, $-\frac{\delta(\alpha_1-\alpha_0)}{\alpha_1-1}$, 0 and $\frac{\delta(\alpha_1-\alpha_0)}{\alpha_1-1}$.*
- *if $\frac{\delta(\alpha_1-\alpha_0)}{\alpha_1-1} > \bar{\imath}$, there exists three multiple equilibria, $-\bar{\imath}$, 0 and $\frac{\delta(\alpha_1-\alpha_0)}{\alpha_1-1}$.*

If the Taylor rule is not kinked, we have two multiple equilibria $-\bar{\imath}$ and 0.

Appendix E. ZLB case: complements

Figure 13 represents the probability to exit the ZLB depending on the value of the Taylor rate $\hat{\imath}_t$ with respect to $-(r^* + \pi^*)$.

Figure 13. Probability to exit ZLB

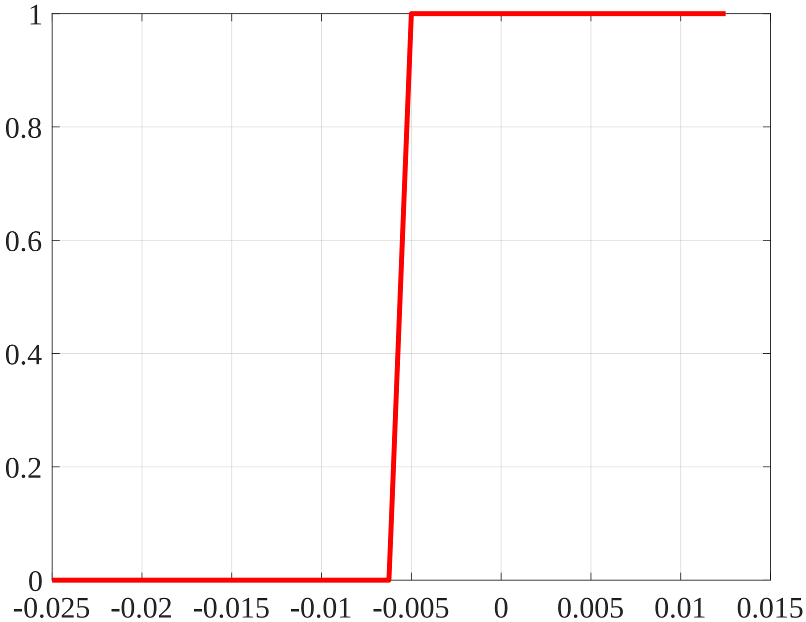


Table 4. Main moments as a function of r^* - case of large shocks

	(α_0, α_1)	$r^* = 0.5\%$	$r^* = 1\%$	$r^* = 2\%$
Frequency ZLB (%)	$(1.1, 1.5)$	18.1	14.5	8.8
	$(1.5, 1.5)$	17.5	13.8	8.1
	$(1.5, 2.1)$	20.5	17.0	11.1
Frequency inside the bands (%)	$(1.1, 1.5)$	25.7	25.8	26.0
	$(1.5, 1.5)$	27.7	27.9	28.2
	$(1.5, 2.1)$	28.9	29.1	29.4
Standard error of annualized inflation (%)	$(1.1, 1.5)$	2.46	2.42	2.37
	$(1.5, 1.5)$	2.32	2.28	2.23
	$(1.5, 2.1)$	2.14	2.11	2.05

E.1. **Characteristics with respect to r^*.** In Table 4, we represent the evolution of the distribution when the steady-state r^* has higher values 1% and 2%, in the case of large shocks. This induces that the time inside the band is small compared to the baseline model. The probability of occurrence of ZLB decreases when r^* increases, but the magnitude of the change remains moderate.

References

Barthélemy, J. and M. Marx (2017): "Solving endogenous regime switching models," *Journal of Economic Dynamics and Control*, 77, 1–25.

——— (2019): "Monetary policy switching and indeterminacy," *Quantitative Economics*, 10, 353–385.

Benhabib, J., S. Schmitt-Grohe, and M. Uribe (2001): "The Perils of Taylor Rules," *Journal of Economic Theory*, 96, 40–69.

Bhattarai, S., J. W. Lee, and W. Y. Park (2014): "Price indexation, habit formation, and the Generalized Taylor Principle," *Journal of Economic Dynamics and Control*, 48, 218–225.

Bianchi, F., L. Melosi, and M. Rottner (2019): "Hitting the Elusive Inflation Target," Working paper series, Federal Reserve Bank of Chicago.

Brainard, L. (2020): "Monetary Policy Strategies and Tools When Inflation and Interest Rates Are Low," Speech at the 2020 U.S. Monetary Policy Forum.

Castelnuovo, E., D. Rodriguez-Palenzuela, and S. Nicoletti Altimari (2003): "Definition of price stability, range and point inflation targets: the anchoring of long-term inflation expectations," Working Paper Series 273, European Central Bank.

Chung, H., B. Doyle, J. Hebden, and M. Siemer (2020): "Considerations Regarding Inflation Ranges," Finance and Economics Discussion Series 2020-075, Board of Governors of the Federal Reserve System (U.S.).

Coeuré, B. (2019): "Monetary policy: lifting the veil of effectiveness," Speech at the at the ECB colloquium on "Monetary policy: the challenges ahead".

Davig, T. and E. M. Leeper (2007): "Generalizing the Taylor Principle," *American Economic Review*, 97, 607–635.

——— (2008): "Endogenous Monetary Policy Regime Change," in *NBER International Seminar on Macroeconomics 2006*, National Bureau of Economic Research, Inc, NBER Chapters, 345–391.

Ehrmann, M. (2021): "Point targets, tolerance bands, or target ranges? Inflation target types and the anchoring of inflation expectations," Working Paper Series 2562, European Central Bank.

Evans, G. (1985): "Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models," *The Quarterly Journal of Economics*, 100, 1217–1233.

Farmer, R. E., D. F. Waggoner, and T. Zha (2009): "Understanding Markov-switching rational expectations models," *Journal of Economic Theory*, 144, 1849 – 1867.

Galí, J. (2015): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications Second edition*, no. 10495 in Economics Books, Princeton University Press.

Grosse-Steffen, C. (2021): "Cross country evidence on quantitative inflation targets," Working paper (forthcoming), Banque de France.

Guerrieri, L. and M. Iacoviello (2015): "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, 70, 22–38.

Knot, K. (2019): "The quest for policy scope: Implications for monetary policy strategies," Speech at at the fourth annual high-level conference Racing for Economic Leadership.

Mishkin, F. S. and N. J. Westelius (2008): "Inflation Band Targeting and Optimal Inflation Contracts," *Journal of Money, Credit and Banking*, 40, 557–582.

Orphanides, A. and V. Wieland (2000): "Inflation zone targeting," *European Economic Review*, 44, 1351–1387.

Rosengren, E. (2018): "Considering Alternative Monetary Policy Frameworks: an Inflation Range With an Adjustable Inflation Target," Speech at the Global Interdependence Center,San Diego.

Rostagno, M., C. Altavilla, G. Carboni, W. Lemke, R. Motto, A. Saint Guilhem, and J. Yiangou (2019): "A tale of two decades: the ECB's monetary policy at 20," Working Paper Series 2346, European Central Bank.

Svensson, L. E. O. (1997): "Inflation forecast targeting: Implementing and monitoring inflation targets," *European Economic Review*, 41, 1111–1146.

Woodford, M. (1986): "Stationary Sunspot Equilibria: The Case of Small Fluctuations around a Deterministic Steady State," *mimeo*.

——— (2011): *Interest and Prices: Foundations of a Theory of Monetary Policy*, Economics Books, Princeton University Press.